J Glob Optim (2010) 47:355–367
DOI 10.1007/s10898-008-9350-6

Quasi-bounded mappings and complementarity problems depending of parameters

G. Isac

Received: 26 August 2008 / Accepted: 26 August 2008 / Published online: 19 September 2008

Abstract In this paper are presented a mathematical tool based on the notion of quasi-bounded mapping, applicable to the study of nonlinear complementarity problems depending of parameters.

Keywords Quasi-bounded mappings · Complementarity problems depending of parameters

1 Introduction

The aim of Complementarity Theory is the study of complementarity problems (linear or nonlinear), from several points of view. Each complementarity problem is related to a particular mathematical model considered *in optimization theory, in the study of equilibrium of economical systems, in mechanics, in engineering etc.* [25–27].

A main chapter in Complementarity Theory is the study of nonlinear complementarity problems, generally, considered in infinite dimensional Hilbert spaces [25–27].

It is well known that there exist many and interesting interactions between Nonlinear Analysis and Nonlinear Complementarity Theory, [22,25–27].

Many nonlinear complementarity problems considered in Physics and in Engineering are nonlinear complementarity problems defined by integral operators with respect to closed convex cones in the space $L^2(\Omega, \mu)$ or in particular Sobolev Hilbert spaces [29,34].

In this paper we present some existence theorems for nonlinear complementarity problems defined by quasi-bounded operators. A. Granas introduced the notions of quasi-bounded operator [20]. It is known that quasi-bounded operators have interesting applications to the study of fixed points for nonlinear mappings [20,36,43]. We consider in this paper nonlinear complementarity problems defined by nonlinear operators, which are quasi-bounded and depending of one or several parameters. This kind of nonlinear complementarity problems,

G. Isac (✉)
Department of Mathematics, Royal Military College of Canada, P.O. Box 17 000, STN Forces, Kingston, ON, Canada K7K 7B4
e-mail: isac-g@rmc.ca

are related to some mathematical models used in Physics, Engineering and in Elasticity. The results presented in this paper can be considered as stimulus for new researches in nonlinear analysis and in particular to obtain new classes of quasi-bounded operators. The reader can also see the books [15,28,31].

2 Preliminaries

We denote by $(E\,\|\cdot\|)$ a Banach space and by $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space. If $(E\,\|\cdot\|)$ is a given Banach space, we denote by $\mathcal{L}(E, E)$ the Banach space of linear continuous mappings from E into E. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. We say that a subset K of E (or of H) is a *pointed convex cone* if K satisfies the following properties:

$$(k_1) K + K \subseteq K;\, (k_2)\lambda K \subseteq K \quad \text{for any} \quad \lambda \in \mathbb{R}_+ \quad \text{and} \quad (k_3)\mathbb{K} \cap (-\mathbb{K}) = \{0\}.$$

If K is a pointed convex cone in a Hilbert space H, then by definition the dual cone of K is:

$$\mathbb{K}^* = \{y \in H \,|\, \langle x, y\rangle \geq 0 \quad \text{for all} \quad x \in \mathbb{K}\}.$$

If K is a closed convex cone in a Hilbert space H, we denote by $P_{\mathbb{K}}$ the metric projection onto K, that is, the mapping $P_{\mathbb{K}}\colon H \to \mathbb{K}$ which associates to any $x \in H$, the unique element in K, denoted by $P_{\mathbb{K}}(x)$ and defined by the condition

$$\|x - P_{\mathbb{K}}(x)\| \leq \|x - y\|, \quad \text{for all} \quad y \in \mathbb{K}.$$

The projection $P_{\mathbb{K}}$ is nonexpansive and therefore it is continuous.

Because $0 \in K$ we have $\|P_{\mathbb{K}}(T(x))\| = \|P_{\mathbb{K}}(T(x)) - P_{\mathbb{K}}(0)\| \leq \|T(x)\|$, for any mapping $T\colon H \to H$, and $x \in H$. If A is a subset of E we denote by $\overline{A}$ the closure of A and by ∂A the boundary of A.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f\colon H \to H$ a mapping. The *general nonlinear complementarity problem* defined by f and K is:

$$NCP(f, \mathbb{K}) : \begin{cases} \text{find } x_0 \in \mathbb{K} \text{ such that} \\ f(x_0) \in \mathbb{K}^* \text{ and } \langle x_0, f(x_0)\rangle = 0. \end{cases}$$

When f is an affine mapping, the problem $NCP(f, K)$ becomes the linear complementarity problem denoted by $LCP(A, b, K)$ where $f(x) = Ax + b$ with $A \in \mathcal{L}(H, H)$ and b is an element in H. For more information about complementarity theory, the reader is referred to [22,25–27].

We consider in this paper the problem $NCP(f, K)$ when the mapping f has one of the following forms:

(i) $f(x) = \rho x - T(x),\ \rho \in \mathbb{R}$ and T is a nonlinear mapping.
(ii) $f(x) = x - \lambda T_1(x) + T_2(x),\ \lambda \in \mathbb{R}$. This form has as particular case the Von Kárman operator i.e., $f(x) = x - \lambda L(x) + T(x),$ with $L \in \mathcal{L}(H, H)$ and T a nonlinear completely continuous operator, homogeneous of degree three. The operator L is completely continuous and self-adjoint [10–14,16,24].
(iii) $f(x) = \rho x - T_1(x) - \varepsilon T_2(x),\ \rho, \varepsilon \in \mathbb{R}$.
(iv) $f(x) = \rho x - \lambda T_1(x) - \varepsilon T_2(x),\ \rho, \lambda, \varepsilon \in \mathbb{R}$.

The kinds of mappings presented above are used in some practical problems, considered in mechanics and in particular in elasticity, [10–14,16,24,34]. In these cases we have complementarity problems depending of parameters. We note that in many practical problems, the operators, T, T_1, T_2 and L used above are integral operators defined on the space L^2.

3 Asymptotic derivatives

Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E, E)$ the Banach space of linear continuous mappings from Einto E. If $f$is a mapping from E into E, the *asymptotic derivative* of f (if it exists) is an element $L \in \mathcal{L}(E, E)$ such that

$$\lim_{\|x\|\to+\infty} \frac{\|f(x) - L(x)\|}{\|x\|} = 0.$$

It is known [3,32,33] that if f has an asymptotic derivative, then it is unique and we denote it by $f^{\infty} = L$. Also in [3] and in [32] it is proved that if f is completely continuous then f^{∞} is completely continuous too. About this result we cite also our paper [23].

Generally, nonlinear integral operators are asymptotically differentiable, under some assumptions. This is the case of Hammerstein or Uryson operators [32,33]. In this sense we cite some facts.

Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded set, which has a piecewise smooth boundary, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $K\colon \Omega \times \Omega \to \mathbb{R}$ two mappings. The Hammerstein operator defined by K, f and the Lebesgue measure is:

$$A(\varphi)(t) = \int_{\Omega} K(t, s)\, f[s, \varphi(s)]\, \mathrm{d}s,$$

where φ is for example in $L^2(\Omega, \mu)$, or in $L^p(\Omega, \mu)$, $(1 < p < \infty)$.

If the following conditions are satisfied:

(1) $\int_{\Omega}\int_{\Omega} K^2(t, s)\, \mathrm{d}t\mathrm{d}s < +\infty$,
(2) *the substitution operator* $f_0(\varphi)(s) = f[s, \varphi(s)]$, where $\varphi \in L^2$ *is such that* $f_0\colon L^2 \to L^2$,
(3) $|f(t, u) - u| \leq \sum S_j(t)\,|u|^{1-p_j} + D(t)$, *where* $t \in \Omega, -\infty < u < +\infty, S_j(t) \in L^{2/p_j}, 0 < p_j < 1, j = 1, 2, \ldots, m$ *and* $D \in L^2$,

then $A : L^2 \to L^2$ *and its asymptotic derivative is*

$$B(\varphi)(t) = \int_{\Omega} K(t, s)\, \varphi(s)\, \mathrm{d}s.$$

The proof of this result is in [33].

We have also the following abstract result. Consider the space $L^2 = L^2(\Omega, \mu)$ and suppose that the operators:

$$A(u)(x) = \int_{\Omega} \mathcal{K}(x, y)\, u(y)\, \mathrm{d}y$$

and

$$f_*(u)(x) = f[x, u(x)]$$

are well defined and are continuous from L^2 into L^2. The abstract form of Hammerstein operator defined by A and f_* is $F(u) = (A \circ f_*)(u)$. If the operator f_* has an asymptotic derivative, denoted by f_*^{∞}, then the operator F is asymptotically derivable and its asymptotic derivative F^{∞} is the linear operator $A \circ f_*^{\infty}$. Indeed, we have

$$\lim_{\|u\|\to\infty} \frac{\left\|F(u) - \left(A \circ f_*^{\infty}\right)(u)\right\|}{\|u\|} = \lim_{\|u\|\to\infty} \frac{\left\|(A \circ f_*)(u) - \left(A \circ f_*^{\infty}\right)(u)\right\|}{\|u\|}$$
$$\leq \lim_{\|u\|\to\infty} \frac{\|(A)\|\,\left\|f_*(u) - f_*^{\infty}(u)\right\|}{\|u\|} = 0$$

In the case of Uryson operator we have the following result.

Consider the Uryson operator i.e.,

$$(Ax)(t) = \int_{\Omega} k(t, s, x(s))\mathrm{d}s,$$

supposed to be well defined from L^2 into L^2.

If the limit

$$\lim_{u\to\infty} u^{-1}k(t, s, u) = k_0(t, s) \quad (t, s \in \Omega),$$

exists, then it is natural to look for an asymptotic derivative A^∞ of the operator A, the following linear integral operator.

$$B(h(t)) = \int_{\Omega} k_0(t, s)\, h(s)\mathrm{d}s.$$

It is known [32] that if $k(t, s, u)$ and $k_0(t, s)\,u$ satisfies some special conditions then the operator A has as asymptotic derivative, the operator B.

We note that the Hyers-Ulam stability of mappings can be also used to find the asymptotic derivative of some mappings [22].

For more information about asymptotic derivatives the reader is referred to [3,32,33].

We consider that the notion of *asymptotic derivative* must be studied in the sense to find new classes of nonlinear operators asymptotically derivable.

4 Quasi-bounded operators

We recall in this section the notion of *quasi-bounded operator* defined by A. Granas in 1962 and used in the fixed-point theory [20]. We recall the definition of quasi-bounded operators in the general case.

Let $(E, \|\cdot\|)$ be a Banach space or a Hilbert space and $f : E \to E$ a mapping.

We say that f is quasi-bounded if and only if $\limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} < +\infty$.

If f is quasi-bounded we denote:

$$[f]_{qb} = \limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} = \inf_{\rho>0} \sup_{\|x\|\geq\rho} \frac{\|f(x)\|}{\|x\|},$$

and we say that $[f]_{qb}$ is *the quasi-norm off.*

We denote $[f]_b = \sup_{x\neq 0} \frac{\|f(x)\|}{\|x\|}$ and if $[f]_b < \infty$ we say that f is *linearly bounded.*

If f is linearly bounded it is known that $[f]_{qb} \leq [f]_b$ and hence, any linearly bounded mapping is quasi-bounded. In particular, any linear continuous mapping from E into E is quasi-bounded and the converse is not true.

Indeed, if $f : E \to E$is a linear continuous mapping then in this case we have

$$\limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} = \inf_{\rho>0} \sup_{\|x\|\geq\rho} \frac{\|f(x)\|}{\|x\|} = \|f\| < +\infty,$$

that is, in this case we have that $[f]_{qb} = \|f\|$.

We note that, the quasi-bounded operators have been considered by several authors in the study of problems related to the fixed-point theory to the subjectivity of nonlinear mappings and to the solvability of nonlinear equations [9,17,20,23,36,39,40,43].

We note that the notion of quasi-bounded operator can be defined also for set-valued mappings.

We say that a mapping $T\colon E \to E$ satisfies condition (BN) (Brezis-Nirenberg) [8]) if

$$\lim_{\|x\| \to +\infty} \frac{\|T(x)\|}{\|x\|} = 0.$$

Obviously, any operator, which satisfies condition (BN), is a *quasi-bounded* operator.

It is easy to prove that if $f\colon E \to E$ has a decomposition of the form $f = h + g$ with h quasi-bounded and g linearly bounded (or satisfying condition (BN) is a quasi-bounded operator.

Also, if $f\colon E \to E$ is bounded, in the sense that there exists $M > 0$ such that $\|f(x)\| \leq M\|x\|$ for any $x \in E$ then f is quasi-bounded. More general, we can prove the following result.

A mapping f: $E \to E$ is quasi-bounded if and only if there exist $\rho > 0$ and two positive constants M_1 and M_2 such that $\|f(x)\| \leq M_1\|x\| + M_2$ for any $x \in E$, with $||x|| > \rho$.

We note also the following examples of quasi-bounded mappings.

If there exists a mapping $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to+\infty} \frac{\varphi(t)}{t} = 0$, then any mapping f: $E \to E$ such that for any $x \in E$, $\|f(x)\| \leq M\|x\| + \varphi(\|x\|)$, with $M > 0$ is a quasi-bounded mapping.

There exists an interesting relation between asymptotic derivability and quasi-boundeness. In this sense we cite the following result.

If f: $E \to E$ has an asymptotic derivative$T \in \mathcal{L}(E, E)$ then f is quasi-bounded and $[f]_{qb} = \|T\|$.

Indeed, we have

$$\begin{aligned}
[f]_{qb} = \limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} &\leq \limsup_{\|x\|\to+\infty} \frac{\|f(x) - T(x)\|}{\|x\|} \\
&+ \limsup_{\|x\|\to+\infty} \frac{\|T(x)\|}{\|x\|} = \limsup_{\|x\|\to+\infty} \frac{\|T(x)\|}{\|x\|} = \|T\|
\end{aligned}$$

and

$$\begin{aligned}
\|T\| = \limsup_{\|x\|\to+\infty} \frac{\|T(x)\|}{\|x\|} &\leq \limsup_{\|x\|\to+\infty} \frac{\|T(x) - f(x)\|}{\|x\|} \\
&+ \limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} = \limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} = [f]_{qb}.
\end{aligned}$$

(We used [7], Chapter 4, Sect. 7, *Proposition 13* and *Corollary 1*).

By this result we obtain that many nonlinear integral operators, under some conditions are quasi-bounded operators, in particular we have this situation for Hammerstein and Uryson operators [32,33].

In 1984, V. H. Weber studied the spectrum of a class of nonlinear operators named φ-asymptotically bounded operators [44].

Let φ be a function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with the property that for a particular $\rho > 0$, $\varphi(t) > 0$ for any $t \geq \rho$.

Wesaythatamapping $f\colon E \to E$ is φ-*asymptoticallybounded* if there exist $b, c \in \mathbb{R}_+$ such that for any x with $b \leq \|x\|$ we have $\|f(x)\| \leq c\varphi(\|x\|)$.

Obviously, if f is φ-asymptotically bounded and φ satisfies condition (BN) then f is quasi-bounded.

The *quasi-norm of a quasi-bounded* mapping has the following properties.

(1) If $f\colon E \to E$ is quasi-bounded and $\lambda \in P$ then $[\lambda f]_{qb} = |\lambda|\,[f]_{qb}$. Indeed, we have $[\lambda f]_{qb} = \limsup_{\|x\|\to+\infty} \frac{\|\lambda f(x)\|}{\|x\|} = |\lambda| \limsup_{\|x\|\to+\infty} \frac{\|\lambda f(x)\|}{\|x\|} = |\lambda|\,[f]_{qb}$.

(2) If $f_1, f_2\colon E \to E$ are quasi-bounded, then $f_1 + f_2$ is quasi-bounded and we have $[f_1 + f_2]_{qb} \leq [f_1]_{qb} + [f_2]_{qb}$.

This formula is a consequence of ([7], Chapter 4, Sect. 7, *Proposition 13*)

We end this section with the following result due to M. A. Krasnoselskii [22,32].

If for a mapping $f\colon E \to E$ there exists $\rho > 0$ (sufficiently big) such that f has a Fréchet derivative, denoted by $f'(x)$, at any point x with $||x|| > \rho$ and $\lim_{\|x\|\to+\infty} \left\| f'(x) - T \right\| = 0$, with $T \in \mathcal{L}(E, E)$, the T is an asymptotic derivative of f.

Consequently, in this case f is quasi-bounded.

5 Applications to complementarity problems depending of parameters

We present in this section some applications of quasi-bounded operators to the study of complementarity problems defined by mappings depending of parameters, like the mappings defined in Sect. 2. We note that complementarity problems or variational inequalities defined by mappings depending of parameters are considered in many practical problems studied in Engineering or in Physics.

About such kinds of problems the reader is referred to [1,6,10–14,16,19,24,29,34].

Our main results presented in this paper are based on the following classical result.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, X a non-empty subset of H and $f\colon X \to H$ a mapping. We recall that f is *compact* if $f(X)$ is a relatively compact subset of H, and we say that f is *completely continuous* if it is continuous and for any bounded set $B \subset X$ we have that $f(B)$ is *relatively compact* in H.

Theorem 1 (Leray-Schauder Alternative) *Let $D \subset H$ be a convex set, U a subset, open in D and such that $0 \in U$. Then each continuous compact mapping $f\colon \overline{U} \to D$ has at least one of the following properties:*

(i) f has a fixed-point, i.e., there exists an element $x_* \in \overline{U}$ such that $f(x_*) = x_*$,
(ii) there exist $x_* \in \partial U$ and $0 < \lambda_* < 1$ *such that* $x_* = \lambda_* f(x_*)$.

Proof A proof of this result is given in [35] □

For more information about Leray-Schauder Alternative the reader is referred to [27].

As application of *Theorem 1* we have the following result.

Theorem 2 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $T\colon H \to H$ a quasi-bounded completely continuous mapping.*

Then for any real number ρ satisfying $[T]_{qb} < \rho$ the problem NCP(f, K), has a solution where $f(x) = \rho x - T(x)$.

Proof We consider the mapping $F(x) = P_{\mathbb{K}}\left[x - \frac{1}{\rho} f(x)\right]$ defined for any $x \in H$. From complementarity theory [22,25,26] we know that the problem $NCP(f, K)$ has a solution, if and only if the mapping F has a fixed-point (which obviously will be in K).

We have that F is quasi-bounded and $[F]_{qb} < 1$. Indeed we have

$$\limsup_{\|x\|\to+\infty} \frac{\|F(x)\|}{\|x\|} = \limsup_{\|x\|\to+\infty} \frac{\left\| P_{\mathbb{K}}\left[\frac{1}{\rho} T(x)\right]\right\|}{\|x\|} \leq \limsup_{\|x\|\to+\infty} \frac{\frac{1}{\rho}\|T(x)\|}{\|x\|}$$
$$= \frac{1}{\rho} \limsup_{\|x\|\to+\infty} \frac{\|T(x)\|}{\|x\|} = \frac{1}{\rho}[T]_{qb} < \frac{1}{\rho}\rho = 1$$

Therefore we have $[F]_{qb} < 1$. Then there exists $r > 0$ such that $\frac{\|F(x)\|}{\|x\|} < 1$ for all $x$with $||x|| \geq r$.

We take $C = H$ and $U = \{x \in H \mid \|x\| < r\}$ and we observe that the assumptions of *Theorem 1* are satisfied.

We take that there is no $x_0 \in \partial U$ and $0 < \lambda_0 < 1$ such that $x_0 = \lambda_0 F(x_0)$, because if we suppose that $x_0 = \lambda_0 F(x_0)$ with $x_0 \in \partial U$ and $0 < \lambda_0 < 1$, we have $\|x_0\| = \lambda_0 \|F(x_0)\| < \|F(x_0)\|$, which is impossible. Therefore applying *Theorem 1* we have that the problem $NCP(f, K)$ has a solution. □

Corollary 3 *If $T: H \to H$ is a quasi-bounded completely continuous mapping such that $T(0) \notin -K^*$ then any real number ρ such that $[T]_{qb} < \rho$ is an eigenvalue for the problem $NCP(f, K)$, i.e., there exists a solution x_* of the problem NCP(f, K) associated to ρ such that $x_* \neq 0$.* □

Theorem 4 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $T_1, T_2: H \to H$, quasi-bounded completely continuous mappings and $K \subset H$ a closed pointed convex cone.*

Consider the mapping $f_{\rho,\varepsilon}: H \to H$ defined by $f_{\rho,\varepsilon}(x) = \rho x - T_1(x) - \varepsilon T_2(x)$, with ρ, ε positive real parameters.

Then for any $\rho > 0$ such that $[T_1]_{qb} < \rho$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ the problem $NCP(f_{\rho,\varepsilon}, \mathbb{K})$ has a solution. Moreover, if $T_1(0) = 0$ and $T_2(0) \notin -\mathbb{K}^*$ or $T_1(0) \notin -\mathbb{K}^*$ and $T_2(0) = 0$, then any $\varepsilon \in]0, \varepsilon_0[$ is an eigenvalue for the problem $NCP(f_{\rho,\varepsilon}, \mathbb{K})$, (i.e., the problem $NCP(f_{\rho,\varepsilon}, \mathbb{K})$ has a nontrivial solution associated to ρ and ε).

Proof Because $[T_1]_{qb} < \rho$, then there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 [T_2]_{qb} < \rho - [T_1]_{qb}$, which implies that for any $\varepsilon \in]0, \varepsilon_0[$ we have $\varepsilon [T_2]_{qb} \leq \varepsilon_0 [T_2]_{qb} < \rho - [T_1]_{qb}$.

Considering the fact that $f_{\rho,\varepsilon}(x) = \rho x - [T_1(x) + \varepsilon T_2(x)]$ we deduce that $T_1 + \varepsilon T_2$ is a completely continuous quasi-bounded mapping, and $[T_1 + \varepsilon T_2]_{qb} \leq [T_1]_{qb} + \varepsilon [T_2]_{qb} < \rho$.

The theorem follows applying *Theorem 2* and obviously, if all the assumptions of our theorem are satisfied we have that any $\varepsilon \in]0, \varepsilon_0[$ is an eigenvalues for the problem $NCP(f_{\rho,\varepsilon}, \mathbb{K})$. □

Now we consider the general case that is the case when the mapping f is dependent of three parameters in the following form:

$$f(x) = \rho x - \lambda T_1(x) - \varepsilon T_2(x).$$

In this case we have the following result.

Theorem 5 *Let $(H, \langle\cdot,\cdot\rangle)$ Hilbert space, T_1, T_2: $H \to H$, quasi-bounded completely continuous mappings and $K \subset H$ a closed pointed convex cone.*

Consider the mapping $f_{\rho,\lambda,\varepsilon}$: $H \to H$ defined by $f_{\rho,\lambda,\varepsilon}(x) = \rho x - \lambda T_1(x) - \varepsilon T_2(x)$, with ρ, λ and ε positive real parameters.

Then for any $\rho > 0, \lambda > 0$ such that $[T_1]_{qb} < \frac{\rho}{\lambda}$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ the problem $NCP\left(f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a solution. Moreover, if $T_1(0) = 0$ and $T_2(0) \notin -\mathbb{K}^*$ or $T_1(0) \notin -\mathbb{K}^*$ and $T_2(0) = 0$, then any $\varepsilon \in]0, \varepsilon_0[$ is an eigenvalue for the problem $NCP\left(f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$, (i.e., the problem $NCP\left(f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a nontrivial solution associated to ρ, λ and ε).

Proof We observe that the problem $NCP\left(f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a solution if and only if the problem $NCP\left(\frac{1}{\lambda} f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a solution. The mapping $\frac{1}{\lambda} f_{\rho,\lambda,\varepsilon}$ has the form

$$\frac{1}{\lambda} f_{\rho,\lambda,\varepsilon}(x) = \frac{\rho}{\lambda} x - T_1 - \frac{\varepsilon}{\lambda} T_2(x),$$

which is the mapping considered in *Theorem 4*, where ρ is replaced by $\frac{\rho}{\lambda}$ and ε by $\frac{\varepsilon}{\lambda}$.

As in the proof of *Theorem 4* there exists $\varepsilon_0 > 0$ such that $\frac{\varepsilon_0}{\lambda}[T_2]_{qb} < \frac{\rho}{\lambda} - [T_1]_{qb}$ which implies that for any $0 < \frac{\varepsilon}{\lambda} < \frac{\varepsilon_0}{\lambda_0}$

$$\left[T_1 + \frac{\varepsilon}{\lambda} T_2\right]_{qb} < [T_1]_{qb} + \frac{\varepsilon}{\lambda}[T_2]_{qb} < \frac{\rho}{\lambda}$$

and applying *Theorem 2* we obtain that the problem $NCP\left(\frac{1}{\lambda} f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a solution for any $\frac{\varepsilon}{\lambda}$ satisfying $0 < \frac{\varepsilon}{\lambda} < \frac{\varepsilon_0}{\lambda}$.

Therefore for any ε such that $0 < \varepsilon < \varepsilon_0$ the problem $NCP\left(f_{\rho,\lambda,\varepsilon}, \mathbb{K}\right)$ has a solution. The last part of the conclusion is evident. □

It is interesting to consider, in particular, the case when f is the Von Kárman operator, that is the case $f(x) = x - \lambda L(x) + T(x)$, where $L \in \mathcal{L}(H, H)$ is self-adjoint and completely continuous and T is a nonlinear completely continuous operator and positive homogeneous of degree three [10–14,24].

We proved in [24] that if $(H, \langle\cdot,\cdot\rangle)$is a Hilbert space, $K \subset H$ is a closed pointed convex cone, T is not necessarily homogeneous and $\langle T(x), x\rangle > 0$ for any $x \in \mathbb{K}\setminus\{0\}$ and $\lambda \in [0, \rho]$, where $\frac{1}{\rho} = \sup_{x\in\mathbb{K}} \frac{\langle L(x),x\rangle}{\|x\|^2}$, then the problem NCP($f, K$) has only the trivial solution.

Now using the quasi-bounded operators we have the following result.

Theorem 6 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone, L: $H \to H$ a nontrivial linear self-adjoint and completely continuous operator. Let T: $H \to H$ be a nonlinear completely continuous operator. Consider the mapping $f(x) = x - \lambda L(x) + T(x)$, where $x \in H$ and $\lambda > \rho$, where $\frac{1}{\rho} = \sup_{x\in\mathbb{K}} \frac{\langle L(x),x\rangle}{\|x\|^2}$. If T is quasi-bounded and $\rho < \frac{1-[T]_{qb}}{\|L\|}$, then for every λ such that $\rho < \lambda < \frac{1-[T]_{qb}}{\|L\|}$, the problem NCP($f, K$) has a solution. Moreover, if we consider a perturbation of f of the form $f_b(x) = x - \lambda L(x) + T(x) + b$, where $b \in H$ and $b \notin \mathbb{K}^*$, then for any λ such that $\rho < \lambda < \frac{1-[T]_{qb}}{\|L\|}$, the problem NCP($f_b, K$) has a nontrivial solution. (We suppose that b has a very small norm).*

Proof Indeed, for any λ such that $\rho < \lambda < \frac{1-[T]_{qb}}{\|L\|}$ we have $\lambda\|L\| < 1 - [T]_{qb}$, which implies $[\lambda L - T]_{qb} \le \lambda\|L\| + [T]_{qb} < 1 - [T]_{qb} + [T]_{qb} = 1$. Applying *Theorem 2*, we have that the problem NCP(f, K) has a solution.

Considering the mapping f_b we observe that the mapping $\lambda L - T - b$ is completely continuous, quasi-bounded and $[\lambda L - T - b]_{qb} \leq \lambda \|L\| + [T]_{qb} < 1 - [T]_{qb} + [T]_{qb} = 1$. Applying again *Theorem 2* we have that the problem NCP(f_b, K) has a solution x_*, and $x_* \neq 0$ because $T(0) = 0$ and $b \notin \mathbb{K}^*$. □

Remark

(1) In the study of elastic plates subjected to unilateral conditions the element b used in Theorem 6 is interpreted as the density of a body force [19].
(2) We note that Theorem 6 is valid if we replace the element b by a mapping A: $H \rightarrow H$ satisfying condition (BN) and we ask that $A(0) \notin \mathbb{K}^*$.

We note that it is interesting to know under what physical or mechanical assumptions the operator T, used in the definition of Won Kárman operator is a quasi-bounded operator.

6 A possible generalization of quasi-boundedness property

Our results presented in the previous section are essentially based on two properties: the complete continuity and the quasi-boundedness.

As we noted above, A. Granas introduce the notion of quasi-bounded mapping as a mathematical tool in the fixed-point theory [20]. After some time this notion has been used to the study of some problems considered in nonlinear analysis [17,36,40,43].

Now, we will follow a similar way that is we will define a generalization of the notion of quasi-bounded mapping, under the name of *B-quasi-bounded mapping* and we will prove a fixed-point theorem for this class of mappings.

Let $(E, \|\cdot\|)$ be a Banach space and $K \subset E$ a closed convex cone (not necessarily pointed). If $\Omega \subset K$ is a non-empty subset we denote by $\overline{\Omega}$, $\partial\Omega$ and $conv(\Omega)$ the closure, the boundary and the convex hull of Ω in K, respectively. Let $\mathcal{P}_b(\mathbb{K})$ the collection of bounded subsets of K.

We say that a function α: $\mathcal{P}_b(\mathbb{K}) \rightarrow [0, +\infty[$ *is a measure of noncompactness* on K if and only if the following properties are satisfied:

(α_1) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact,
(α_2) $\alpha(A) = \alpha\left(\overline{A}\right)$,
(α_3) $A_1 \subseteq A_2$ implies $\alpha(A_1) \leq \alpha(A_2)$,
(α_4) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
(α_5) $\alpha(\lambda A) = \lambda\alpha(A)$ for any $\lambda \in \mathbb{R}_+$,
(α_6) $\alpha(conv(A)) = \alpha(A)$,
(α_7) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

We cite as example of a measure of noncompactness, the Kuratowski measure of noncompactness defined by:

$$\alpha(A) = \inf\{r > 0 \,|\, A \text{ admits a finite cover by sets of diameter at most } r\}.$$

About measures of noncompactness the reader is referred to [2,4,5,41].

Let Ω be a nonempty subset of K and α a measure of noncompactness on K. We say that a continuous mapping f: $\Omega \rightarrow \mathbb{K}$ is said to be countably α-condensing if $\alpha(f(D)) < \alpha(D)$ for each countably bounded set $D \subseteq \Omega$ with $\alpha(D) > 0$. M. Väth defined the notion of countably α-condensing mapping (see [18,42,41] and their references).

The fact that in this notion we consider only countably bounded sets is important in connections with differential and integral operators of vector functions in nonseparable spaces [21,37,38,42,41].

A continuous homotopy h: $[0, 1] \times \Omega \to \mathbb{K}$ *is said to be countably* α-condensing if $\alpha\left(h\left([0,1] \times D\right)\right) < \alpha\left(D\right)$ for each countably bounded set $D \subseteq \Omega$ with $\alpha(D) > 0$.

We need to recall the properties of a topological index for countably α-condensing mappings defined by M. Väth in [41].

Let Ω be a non-empty bounded open set in K. If f: $\overline{\Omega} \to \mathbb{K}$ is a countably α-condensing mapping, without fixed pointes on $\partial\Omega$, then as it is presented in [41], it is well defined a fixed point index of f with respect to Ω. This fixed-point index is denoted by $ind_{\mathbb{K}}\left(f, \Omega\right)$ and it is an integer.

We need to cite the following of its properties.

Proposition 7 *Let Ω be a non-empty bounded open set in $\mathbb{K}$ and f: $\overline{\Omega} \to K$ a countably α-condensing mapping such that f has no fixed point on $\partial\Omega$. Then the following properties of $ind_{\mathbb{K}}\left(f, \Omega\right)$ are satisfied:*

(1) Existence: If $ind_{\mathbb{K}}\left(f, \Omega\right) \neq 0$ then f has a fixed-point in Ω,
(2) Normalization: If $f \equiv 0$ and $0 \in \Omega$ then $ind_{\mathbb{K}}\left(f, \Omega\right) = 1$.
(3) Homotopy invariance: If h: $[0, 1] \times \overline{\Omega} \to \mathbb{K}$ is a countably α-condensing homotopy such that $h\left(t, x\right) \neq x$ for all $\left(t, x\right) \in [0, 1] \times \partial\Omega$ then $ind_{\mathbb{K}}\left(h\left(0, \cdot\right), \Omega\right) = ind_{\mathbb{K}}\left(h\left(1, \cdot\right), \Omega\right)$.

Proof For the proofs of properties indicated above the reader is referred to ([41], Theorem 1.3 and Corollary 2.1). □

Now we suppose given a mapping B: $E \times E \to P$ satisfying the following properties:

(b_1) $B\left(\lambda x, y\right) = \lambda B\left(x, y\right)$ for any $\lambda \in \mathbb{R}_+ \setminus \{0\}$ *and any* $x, y \in E$,
(b_2) $B(x, x) > 0$ *for any* $x \in E \setminus \{0\}$.

The reader can find examples of mappings B satisfying properties (b_1), (b_2) in [30]. In particular any semi-inner product in Lumer's sense satisfies (b_1) and (b_2)

Definition 1 We say that a mapping f: $E \to$ E is B-quasi-bounded with respect to a closed convex cone $K \subseteq$ E if and only if

$$\limsup_{\substack{\|x\| \to +\infty \\ x \in K}} \frac{B\left(f\left(x\right), x\right)}{B\left(x, x\right)} < +\infty.$$

If f is B-quasi-bounded we denote $[f]^{\mathbb{K}}_{Bqb} = \limsup_{\substack{\|x\| \to +\infty \\ x \in \mathbb{K}}} \frac{B(f(x),x)}{B(x,x)}$.

If $K = E$, we denote $[f]^{\mathbb{K}}_{B_{qb}}$ by $[f]_B$. If E is a Hilbert space $(H, \langle\cdot, \cdot\rangle)$ and if B is the inner-product $\langle\cdot, \cdot\rangle$, then in this case we say that a mapping f: $E \to E$ is scalarly-quasi-bounded (S-quasi-bounded) if f is $\langle\cdot, \cdot\rangle$-quasi-bounded with respect to K (or with respect to E). It is easy to observe that on a Hilbert space, taking $B = \langle\cdot, \cdot\rangle$ we have that any quasi-bounded mapping f: $H \to H$ in the classical sense, is S-quasi-bounded. If B is a semi-inner-product in Lumer's sense on a Banach space E, then any quasi-bounded mapping in the classical sense is B-quasi-bounded, if the semi-inner-product is compatible with the norm of E.

More general if $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ are two Banach spaces and we have that a mapping B: $E \times F \to \mathbb{R}$ satisfies the properties (b_1) and (b_2), we can say that a mapping f: $E \to F$ is B-quasi-bounded on E if $\limsup_{\|x\| \to +\infty} \frac{B(x, f(x))}{B(x,x)} < +\infty$.

Obviously, this definition can be considered also with respect to a closed convex cone $K \subset E$.

We have the following result.

Theorem 8 *Let* $(E, \|\cdot\|)$ *be a Banach space,* $K \subset E$ *a closed convex cone and* $f\colon E \to E$ *a mapping.*

If the following assumptions are satisfied:

(1) f *is countably* α*-condensing,*
(2) $f(K) \subseteq K$*,*
(3) f *is B-quasi-bounded and* $[f]^{\mathbb{K}}_{Bqb} < 1$*, then f has a fixed point in* K*.*

Proof Because $\limsup\limits_{\substack{\|x\|\to+\infty \\ x\in\mathbb{K}}} \frac{B(f(x),x)}{B(x,x)} < 1$, then there exist $\rho > 0$, $0 < \rho < 1$ and $r > 0$ such that $\frac{B(f(x),x)}{B(x,x)} \le \rho$ for any x with $\|x\| \ge r$.

Therefore we have $B\left(f\left(x\right), x\right) \le \rho B\left(x, x\right)$, for all x with $||x|| \ge r$. We consider the set $B^r_{\mathbb{K}} = \{x \in \mathbb{K} \mid \|x\| < r\}$. Obviously $0 \in B^r_{\mathbb{K}}$ and $\partial B^r_{\mathbb{K}} = \{x \in \mathbb{K} \mid \|x\| = r\}$.

Let h: $[0, 1] \times \overline{B^r_{\mathbb{K}}} \to \mathbb{K}$ be the continuous homotopy defined by

$$h\left(t, x\right) := tf\left(x\right) \text{ for any } \left(t, x\right) \in [0, 1] \times \overline{B^r_{\mathbb{K}}}.$$

We show that h is countably condensing. Indeed let D be a countably not precompact subset in $\overline{B^r_{\mathbb{K}}}$, with $\alpha(D) > 0$.

Since, $h\left([0, 1] \times D\right) = \bigcup\limits_{0\le t\le 1} tf\left(D\right) \subseteq conv\left[f\left(D\right) \cup \{0\}\right]$, then applying the properties of the measure of noncompactness α we have

$$\alpha\left(h\left([0, 1] \times D\right)\right) = \alpha\left(\bigcup_{0\le t\le 1} tf\left(D\right)\right) \le \alpha\left(f\left(D\right)\right) < \alpha\left(D\right).$$

Therefore, h is countably α-condensing. We have also that

$$h\left(t, x\right) \ne x \text{ for all } \left(t, x\right) \in [0, 1] \times \partial B^r_{\mathbb{K}}.$$

Indeed, if $h(t, x) = x$ for some $(t, x) \in [0, 1] \times \partial B^r_{\mathbb{K}}$then (considering the definition of h we have $t \ne 0$. Considering the fact that $||x|| = r$ and $0 < t \le 1$, we have

$$B\left(x, x\right) = B\left(tf\left(x\right), x\right) = tB\left(f\left(x\right), x\right) \le t\rho B\left(x, x\right) \le \rho B\left(x, x\right),$$

which implies, $B\left(x, x\right)[1 - \rho] \le 0$ and finally $1 \le \rho$, but this is impossible, because ρ was selected such that $0 < \rho < 1$.

Therefore, the homotopy invariance of the fixed-point index implies

$$ind_{\mathbb{K}}\left(h\left(1, \cdot\right), B^r_{\mathbb{K}}\right) = ind_{\mathbb{K}}\left(h\left(0, \cdot\right), B^r_{\mathbb{K}}\right).$$

Since $h(0,\cdot) \equiv 0$ and $0 \in B^r_{\mathbb{K}}$, from *Proposition* 7 we have that f has a fixed point in $B^r_{\mathbb{K}}$ and the proof is complete. □

Corollary 9 *If* $(H, \langle\cdot, \cdot\rangle)$ *is a Hilbert space,* $K \subset H$ *a closed convex cone, f:* $K \to K$ *a countably* α*-condensing mapping and*

$$\limsup_{\substack{\|x\|\to+\infty \\ x\in\mathbb{K}}} \frac{\langle f\left(x\right), x\rangle}{\|x\|^2} < 1,$$

then f has a fixed point in K*.*

Remark We note that Theorem 8 is a generalization of ([30], *Theorem 6.1*) but the proof presented above is different.

We have also the following application to the study of complementarity problems.

Corollary 10 *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone. Let* f*:* $H \to H$ *be a mapping such that* $f(x) = x - \mu A(x) - \lambda T(x)$*, for any* $x \in H$*, where* A*,* T*:* $H \to H$ *and* $\mu, \lambda \in \mathbb{R}_+ \setminus \{0\}$. *The mapping A is supposed to satisfy condition (BN) and T is quasi-bounded with* $[T]_{qb} > 0$*.*

Then for any μ and λ such that $\lambda < \frac{1}{[T]_{qb}}$ and the mapping $\Phi(x) = P_{\mathbb{K}}[\mu A(x) + \lambda T(x)]$ is countably α-condensing, the problem NCP(f, K) has a solution.

Proof We apply Theorem 8 taking $f = \Phi$ and $B(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. □

Remark Corollary 10 can be applied to the study of some nonlinear complementarity problems defined by integral operators and depending of parameters.

Comments We proposed in this paper some mathematical tools, based on quasi-bounded operators applicable to the study of nonlinear complementarity problems depending of parameters. Certainly other developments of this subject are possible.

Acknowledgements This research was supported by the Mexico National Council on Science and Technology CONACyT-SEP project SEP-2004-C01-45787.

References

1. Adly, S., Goeleven, D., Théra, M.: Recession mapping and noncoercive variational inequalities. Nonlinear Anal. Theory Meth. Appl. **26**(9), 1573–1603 (1996). doi:10.1016/0362-546X(94)00364-N
2. Akhmerov, R.R., Kamenskii, M.I., Potapov, A.S., Rodkina, A.E., Sadovskii, B.N.: Measures of Noncompactness and Condensing Operators. Birkhäuser Verlag (1992)
3. Amann, H.: Lecture on Some Fixed Point Theorems, IMPA. Rio de Janeireo-GB (1974)
4. Ayerbe Toledano, J.M., Dominguez Benavides, T., Lopez Acedo, G.: Measure of Noncompactness in Metric Fixed Point Theory. Birkhäuser Verlag (1997)
5. Banas, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Marcel Dekker Inc. (1980)
6. Bezine G., Cimetière A., Gelbert J.P.: Unilateral buckling of thin elastic plates by the boundary integral equation methods. Int. J. Numer. Meth. Eng. **21**, 2189–2199 (1985). doi:10.1002/nme.1620211206
7. Bourbaki, N.: Topologie Générale. Hermann, Paris (1960)
8. Brezis, H., Nirenberg, L.: Characterization of the ranges of some nonlinear operators and applications to boundary value problems. Ser. IV. Ann. Scuola Normale Superiore Pisa, Classe di Scienze **2**, 225–326 (1978)
9. Cain, G.L. Jr., Nashed, M.Z.: Fixed points and stability for a sum of two operators in locally convex spaces. Pac. J. Math. **39**(3), 581–592 (1971)
10. Ciarlet, P., Rabier, P.: Les Equations de Von Kárman. Lecture Notes in Math., Nr. 826. Springer-Verlag (1980)
11. Cimetière A.: Méthode de Galerkin pour flambement de plaques. C. R. Sc. Paris, t. 284, Série A, 1307–1310 (1977)
12. Cimetière, A.: Flambement unilateral d'une plaque reposant sans frottement sur un support élastique tridimensionnel. C. R. Acad. Sc. Paris, t. 290, Serie B, 337–340 (1980)
13. Cimetière, A.: Un problème de flambement unilateral en théorie de plaques. J. Mecanique **19**(1), 183–202 (1980)
14. Cimetière, A.: Méthode de Liapounov-Schmidt et branche de bifurcation pour une class d'inéquations variationnelles. C. R. Acad. Sc. Paris, t. 30, Serie (I), Nr. **15**, 565–568 (1985)
15. Cottle, R., Pang, J., Stone, R.: The Linear Complementarity Problem. Academic Press, Boston (1992)
16. Do, M.C.: Bifurcation theory for elastic plates subjected to unilateral conditions. J. Math. Anal. Appl. **60**, 435–448 (1977). doi:10.1016/0022-247X(77)90033-6

17. Furi, M., Vignoli, A.: A nonlinear spectral approach to surjectivity in Banach spaces. J. Funct. Anal. **20**, 304–318 (1975). doi:10.1016/0022-1236(75)90037-3
18. Giorgieri, E., Väth, M.: A characterization of 0-epi Maps with a Degree. Preprint, University of Rome "Tor Vergata"
19. Goeleven, D., Nguen, V.H., Théra, M.: Nonlinear eigenvalues problems governed by a variational inequality of Von Kárman's type: a degree theoretic approach. Topol. Meth. Nonlienar Anal. **2**, 253–276 (1993)
20. Granas, A.: The theory of compact vector fields and some of its applications to topology of functional spaces, (I). Rozpr. Mathematyczne **XXX**, 1–93 (1962)
21. Heinz, H.P.: On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions. Nonlinear Anal. **7**(12), 1351–1371 (1983). doi:10.1016/0362-546X(83)90006-8
22. Hyers, D.H., Isac, G., Rassias, T.M.: Topics in Nonlinear Analysis and Applications. World Scientific, Singapore (1997)
23. Isac, G.: Opérateurs asymptotiquement linéaires sur des espaces localement convexes. Colloquium Math. **XLVI**(1), 67–72 (1982)
24. Isac, G.: Nonlinear complementarity problem and Galerkin method. J. Math. Anal. Appl. **108**(2), 563–575 (1985). doi:10.1016/0022-247X(85)90045-9
25. Isac,G.: Complementarity Problems. Lecture Notes in Mathematics, Nr. 1528. Springer-Verlag (1992)
26. Isac, G.: Topological Methods in Complementarity Theory. Kluwer Academic Publishers (2000)
27. Isac, G.: Leray-Schauder Type Alternatives, Complementarity Problems and Variational Inequalities. Kluwer Academic Publishers (2006)
28. Isac, G.: Asymptotic Derivable Fields and Nonlinear Complementarity Problems. Preprint (2007)
29. Isac, G., Motreanu, D.: On the solvability of complementarity problems and variational inequalities with integral operators. J. Nonlinear Convex Anal. **4**(3), 333–351 (2003)
30. Isac, G., Nemeth, S.Z.: Scalar derivatives and scalar asymptotic derivatives. An Altman type fixed-point theorem on convex cones and some applications. J. Math. Anal. Appl. **290**, 452–468 (2004). doi:10.1016/j.jmaa.2003.10.030
31. Isac, G., Rassias, T.M.: On the Hyers-Ulam stability of ψ-additive mappings. J. Approx. Theory **72**(2), 131–137 (1993). doi:10.1006/jath.1993.1010
32. Krasnoselskii, M.A.: Topological Methods in the Theory of Nonlinear Integral Equations (in Russian), Gostekhizdat, Moscow (1956). Engl. Transl., Macmillan, New York (1964)
33. Krasnoselskii, M.A.: Positive Solutions of Operator Equations. Noordhoff, Groningen (1964)
34. Le, V.K., Schmitt, K.: Global Bifurcation in Variational Inequalities. Applications to Obstacles and Unilateral Problems. Springer (1997)
35. Leray, J., Schauder, J.: Topologie et equations fonctionnelles. Ann. Sc. Ecole. Norm. Sup. **51**, 45–78 (1934)
36. Martelly, M., Vignoli, A.: Some surjectivity results for non compact multivalued maps. Red. Acad. Sci. Fis. Mater. **41**(4), 57–66 (1974)
37. Mönch, H.: Boundary value problems for nonlinear ordinary differential equations in Banach spaces. Nonlinear Anal. **4**(5), 985–999 (1980). doi:10.1016/0362-546X(80)90010-3
38. Mönch, H., Von Harten, G.F.: On the Cauchy problem for ordinary differential equations in Banach spaces. Arch. Math. Basel **39**, 153–160 (1982). doi:10.1007/BF01899196
39. Petryshyn, W.V.: Remarks on condensing and k-set contractive mappings. J. Math. Anal. Appl. **39**, 717–741 (1972). doi:10.1016/0022-247X(72)90194-1
40. Schechter, M., Shapiro, J., Snow, M.: Solution of the nonlinear problem $\mathrm{AU} = N(U)$ in a Banach space. Trans. Am. Math. Soc. **241**, 69–78 (1978). doi:10.2307/1998833
41. Väth, M.: Fixed-point theorems and fixed point index for countably condensing maps. Topol. Meth. Nonlinear Anal. **13**, 341–363 (1999)
42. Väth M.: Volterra and Integral Equations of Vector Functions. Marcel Dekker, New York (2000)
43. Vignoli, A.: On qusi-bounded mappings and nonlinear functional equations. Atti. Accad. Naz. Lincei, VIII Ser. Rend. Cl. Sci Fis. Mater. Nat. **50**, 114–117 (1971)
44. Weber, V.H.: Φ-Asymptotisches spectrum und surjektivitätssätze vom Fredholm type fur michtlineare operatoren mit anwendungen. Math. Nach. **117**, 7–35 (1984)